Heegaard Floer homology and Dehn surgery

Problem Set 1

Problem 1. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be $n$ mutually disjoint, simple closed curves on a closed oriented surface $\Sigma$. Prove that the homology classes $[\alpha_1], \ldots, [\alpha_n] \in H_1(\Sigma)$ are linearly independent if and only if the complement $\Sigma \setminus (\alpha_1 \cup \cdots \cup \alpha_n)$ is connected.

Problem 2. Find a genus 1 Heegaard diagram of $S^3$, and use it to compute $HF^\infty(S^3), HF^-(S^3), HF^+(S^3)$.

Problem 3. Let

$$(\Sigma, \{\alpha_1, \ldots, \alpha_g\}, \{\beta_1, \ldots, \beta_g\})$$

be a Heegaard diagram of $Y$. Prove

$$H_1(Y) \cong H_1(\Sigma)/([\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g]).$$

Problem 4. Prove the map $\delta: \text{Spin}^c(Y) \to H^2(Y)$ is a one-to-one correspondence.

Problem 5. Suppose $s_1, s_2 \in \text{Spin}^c(Y)$, prove

$$\delta(s_1, s_2) = -\delta(s_2, s_1), \quad c_1(s_1) - c_1(s_2) = 2\delta(s_1, s_2).$$

As a consequence, show that the map $c_1: \text{Spin}^c(Y) \to H^2(Y)$ is injective if $H_1(Y)$ has no 2–torsion.
CODIMENSION ONE FOLIATIONS - PROBLEM SET 1
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(1) Let $\mathcal{F}_m$ be the foliation of $T^2$ described in Lecture 1.

(a) Prove that if $m \in \mathbb{Q} \cup \left\{ \frac{1}{2} \right\}$, then the leaves of $\mathcal{F}_m$ are simple closed curves.

(b) Prove that if $m \notin \mathbb{Q} \cup \left\{ \frac{1}{2} \right\}$, then the leaves of $\mathcal{F}_m$ are injectively immersed copies of $\mathbb{R}$, and any leaf of $\mathcal{F}_m$ is dense in $T^2$.

(c) What can you say about the leaf space of $\mathcal{F}_m$?

(2) An alternate way of viewing $\mathcal{F}_m$: as a suspension of a homeomorphism of $S^1$:

Form a foliation $\mathcal{G}_\phi$ of $T^2$ as follows. First foliate $[0, 1] \times S^1$ by straight line segments $[0, 1] \times \{ t \}$. Then glue $\{ 1 \} \times S^1$ to $\{ 0 \} \times S^1$ by the homeomorphism $\phi : \{ 1 \} \times S^1 \rightarrow \{ 0 \} \times S^1 : (1, e^{2\pi it}) \mapsto (0 , e^{2\pi imt})$.

Let $\gamma$ denote the simple closed curve $\{ 1 \} \times S^1$. Revisit the questions of (1a) and (1b) with this model of $(T^2, \mathcal{F}_m)$ in mind.

(3) Denjoy blow-up and Denjoy splitting:

If necessary, consult Wikipedia on the Cantor function $c : [0, 1] \rightarrow [0, 1]$ before proceeding further. Recall that the Cantor function is continuous. Note that under the identification $S^1 = [0, 1]/ \sim$, where $0 \sim 1$, there is a “Cantor function” $c : S^1 \rightarrow S^1$.

Suppose $m$ is irrational, and view $\mathcal{F}_m$ as given by a suspension of $\phi$ as given in (2). The Denjoy blowup and Denjoy splitting of $\mathcal{F}_m$ along a leaf $L$ is described as follows.

Label the leaves of $\mathcal{F}$ by $L_x$, where $x \in L \cap \gamma$. Pick any $x \in S^1$, and let $x_n = \phi^n(x)$ for all $n \in \mathbb{Z}$. (Equivalently, pick any leaf $L$ of $\mathcal{F}_m$ and enumerate the countable many points $L \cap \gamma$ as $x_n, n \in \mathbb{Z}$.)

Make the following precise:

(a) “Blow up” $S^1$ to a longer circle $C$ by replacing each $x_n$ by a compact interval $J_n$, where the sum of the lengths of the intervals $J_n$ is finite.

(b) Define $\psi : C \rightarrow C$ to agree with $\phi$ on the complement of the orbit $x_n$, and extend linearly over the interior of the intervals $J_n$.

The foliation $\mathcal{G}_\psi$ is called the Denjoy blowup of $\mathcal{F}_m = \mathcal{G}_\phi$ along $L_x$. Removing the leaves of $\mathcal{G}_\psi$ passing through the interiors of the the $J_n$ results in a foliation $\mathcal{F}'_m$ obtained by Denjoy splitting $\mathcal{F}_m$ open along the leaf $L_x$. Note that the transverse cross-section $\mathcal{F}'_m \cap \gamma$ is a Cantor set.

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Let $\tau$ be a train track, and $N(\tau)$ an I-fibered regular neighbourhood of $\tau$. Recall that a curve is \textit{carried by} $\tau$ if it can be isotoped to lie in $N(\tau)$ everywhere transverse to the I-fibering. It is \textit{fully carried by} $\tau$ if it is carried, with nonempty intersection with each I-fiber. A foliation is carried (respectively, fully carried) by $\tau$ if after Denjoy splitting along finitely many leaves, the resulting lamination can be isotoped so that every leaf is carried (fully carried) by $\tau$.

Identify which foliations $\mathcal{F}_m$ are carried (respectively, fully carried) by the train tracks $\tau$ shown below. Do these train tracks carry (respectively, fully carry) any other foliations of $T^2$?